Rewarding Learning

General Certificate of Secondary Education

Centre Number

Candidate Number

Further Mathematics

Unit 1 **(With calculator)**
Pure Mathematics

[GFM11]

GFM11

Assessment

Assessment Level of Control:
Tick the relevant box (✓)

Controlled Conditions	
Other	

TIME

2 hours.

INSTRUCTIONS TO CANDIDATES

Write your Centre Number and Candidate Number in the spaces provided at the top of this page.
You must answer the questions in the spaces provided.
Do not write outside the boxed area on each page.
Complete in black ink only. **Do not write with a gel pen.**
All working **must** be clearly shown in the spaces provided. Marks may be awarded for partially correct solutions.
Where rounding is necessary give answers correct to **2 decimal places** unless stated otherwise.
Answer **all fourteen** questions.

INFORMATION FOR CANDIDATES

The total mark for this paper is 100.
Figures in brackets printed down the right-hand side of pages indicate the marks awarded to each question or part question.
You may use a calculator.
The Formula Sheet is on page 2.

12399.03 **R**

Formula Sheet

PURE MATHEMATICS

Quadratic equations: If $ax^2 + bx + c = 0$ $(a \neq 0)$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Differentiation: If $y = ax^n$ then $\dfrac{dy}{dx} = nax^{n-1}$

Integration: $\int ax^n \, dx = \dfrac{ax^{n+1}}{n+1} + c$ $(n \neq -1)$

Logarithms: If $a^x = n$ then $x = \log_a n$

$\log (ab) = \log a + \log b$

$\log \left(\dfrac{a}{b}\right) = \log a - \log b$

$\log a^n = n \log a$

Matrices: If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $\det \mathbf{A} = ad - bc$

and $\mathbf{A}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $(ad - bc \neq 0)$

1 Find $\frac{dy}{dx}$ if $y = 6x^3 - \frac{4}{3x^3} + 2x$ $= 6x^3 - \frac{4}{3}x^{-3} + 2x$

$\frac{dy}{dx} = 18x^2 + 4x^{-4} + 2$

$= 18x^2 + \frac{4}{x^4} + 2$

Answer ______________________________ [3]

2 Find $\int \left(\frac{x^3}{4} - \frac{1}{x^2} + 5 \right) dx$ $= \int \left(\frac{1}{4}x^3 - x^{-2} + 5 \right) dx$

$= \frac{1}{16}x^4 + x^{-1} + 5x + c$

$= \frac{x^4}{16} + \frac{1}{x} + 5x + c$

Answer ______________________ [4]

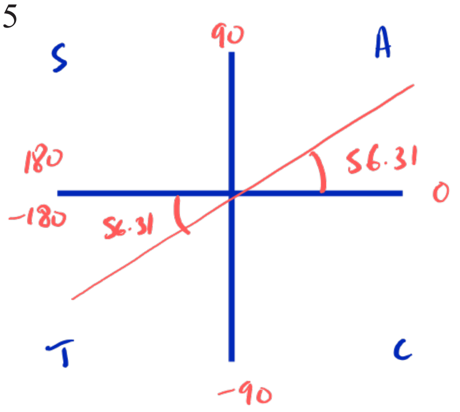
3 **(i)** Solve the equation

$$\tan x = 1.5$$

for $-180° \leqslant x \leqslant 180°$

$\tan x = 1.5$

$\tan^{-1}(1.5) = 56.31°$



Answer $56.31°, -123.69°$ [2]

(ii) **Hence** solve the equation

$$\tan(3\theta - 10°) = 1.5$$

for $-60° \leqslant \theta \leqslant 60°$

$(3\theta - 10) = 56.31°$

$3\theta = 66.31$

$\theta = 22.1°$

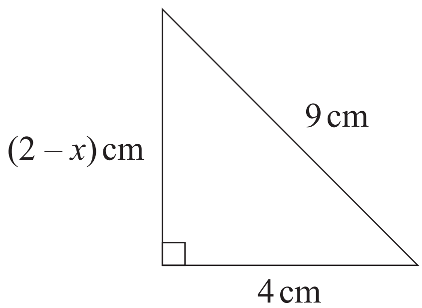
$3\theta - 10 = -123.69$

$3\theta = -113.69$

$\theta = -37.9°$

Answer $22.1°, -37.9°$ [3]

4 The lengths of the sides of a right-angled triangle are 9 cm, 4 cm and $(2 - x)$ cm as shown in the diagram below.



(i) Show that $x^2 - 4x - 61 = 0$

$(2-x)^2 + 4^2 = 9^2$

$(2-x)(2-x) + 4^2 = 9^2$

$4 - 4x + x^2 + 16 = 81$

$x^2 - 4x - 61 = 0$

[2]

(ii) Using the method of **completing the square**, find the **value** of x, giving your answer in surd form.

$x^2 - 4x - 61 = 0$

$(x-2)^2 - (-2)^2 - 61 = 0$

$(x-2)^2 - 4 - 61 = 0$

$(x-2)^2 = 65$

$x - 2 = \sqrt{65}$

$x = 2 \pm \sqrt{65}$

Length $x-2$ negative if $2+\sqrt{65}$

Answer $x = 2 - \sqrt{65}$ [4]

5 Solve the inequality

$$x(2x - 1) - 3 > 0$$

You **must** show clearly each stage of your solution.

$2x^2 - x \overset{-6}{-3} > 0$

$\left.\begin{array}{l} (\times) \to -6 \\ (+) \to -1 \end{array}\right\} -3, 2$

$2x^2 + 2x - 3x - 3 > 0$

$2x(x+1) - 3(x+1) > 0$

$(2x-3)(x+1) > 0$

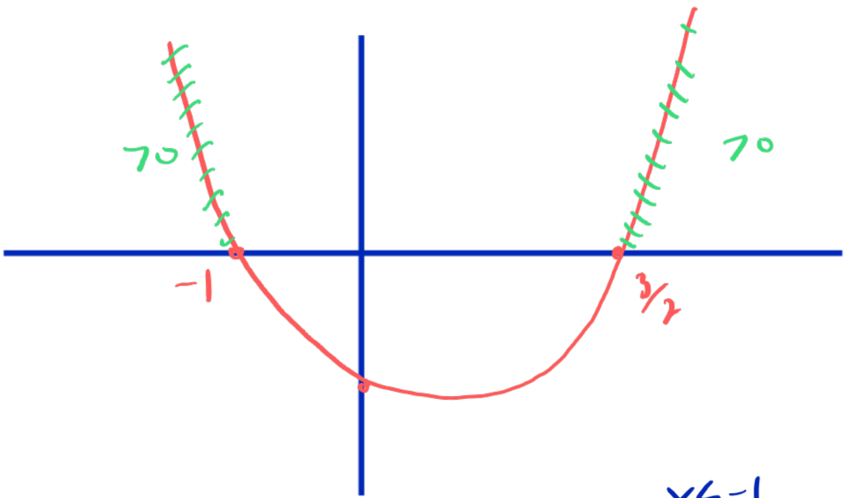
Roots

$2x - 3 = 0$

$x = 3/2$

$x + 1 = 0$

$x = -1$



Answer $x < -1$, $x > 3/2$ [5]

6 Matrices **P**, **Q** and **R** are defined by

$$\mathbf{P} = \begin{bmatrix} x & -4 \\ 6 & y \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

(i) If **PQ** = **R**, find the value of x.

$$\begin{bmatrix} 2(x) - 4(0) \\ 6(2) + y(0) \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 2x \\ 12 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \end{bmatrix}$$

Answer x = -3 [2]

(ii) **Hence**, if **P** has no inverse, find the value of y.

No inverse → det = 0

$$xy = -4(6)$$

$$-3y = -24$$

$$y = 8$$

Answer y = ______________ [2]

[Turn over

7 **(a)** Solve the equation

$$4^{3x-2} = 6^{x-1}$$

$$\log 4^{3x-2} = \log 6^{x-1}$$

$$(3x-2)\log 4 = (x-1)\log 6$$

$$3x\log 4 - 2\log 4 = x\log 6 - \log 6$$

$$3x\log 6 - x\log 6 = 2\log 4 - \log 6$$

$$x(3\log 6 - \log 6) = 2\log 4 - \log 6$$

$$x = \frac{2\log 4 - \log 6}{3\log 6 - \log 6}$$

Answer 0·41 [5]

(b) If $y = \log 4$ and $z = \log\left(\frac{1}{2}\right)$, write y in terms of z.

$z = \log \frac{1}{2}$

$= \log 2^{-1}$

$= -\log 2$

$y = \log 2^2$

$= 2\log 2$

$= -2z$

Answer $y =$ $-2z$ [2]

[Turn over

8 Simplify **fully** the following algebraic expressions.

(i) $$\frac{x^2 - 5x + 6}{x^2 - 9x + 18} \div \frac{x - 2}{2x^2 - 12x}$$

$$\frac{\cancel{(x-3)}\cancel{(x-2)}}{\cancel{(x-6)}\cancel{(x-3)}} \times \frac{2x\cancel{(x-6)}}{\cancel{(x-2)}}$$

$$= 2x$$

Answer ______________________________ [5]

(ii) $\dfrac{2x}{x-1} - \dfrac{x+1}{x^2-x}$

$$\frac{2x}{x-1} - \frac{x+1}{x(x-1)}$$

$$\frac{2x^2}{x(x-1)} - \frac{x+1}{x(x-1)}$$

$$\frac{2x^2-(x+1)}{x(x-1)}$$

$$\frac{2x^2-x\overset{-2}{-1}}{x(x-1)}$$

$\times \to -2$
$+ \to -1$ $\}$ $-2, 1$

$$\frac{2x^2-2x+x-1}{x(x-1)} = \frac{2x(x-1)+1(x-1)}{x(x-1)}$$

$$= \frac{(2x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

Answer $\dfrac{2x+1}{x}$ [5]

[Turn over

9 A curve is defined by the equation $y = -x^2 + 3x + 4$

(i) Find the **coordinates** of the points where the curve crosses the x-axis.

$y=0$

$-x^2+3x+4=0$

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$x=-1 \quad , \quad 4$

$y=0 \quad , \quad 0$

Answer $(-1, 0) \quad (4, 0)$ [2]

(ii) Find the coordinates of the turning point of the curve.

$\frac{dy}{dx} = -2x+3 = 0$

$2x = 3$

$x = 3/2$

$y = -(3/2)^2 + 3(3/2) + 4 = 6\frac{1}{4}$

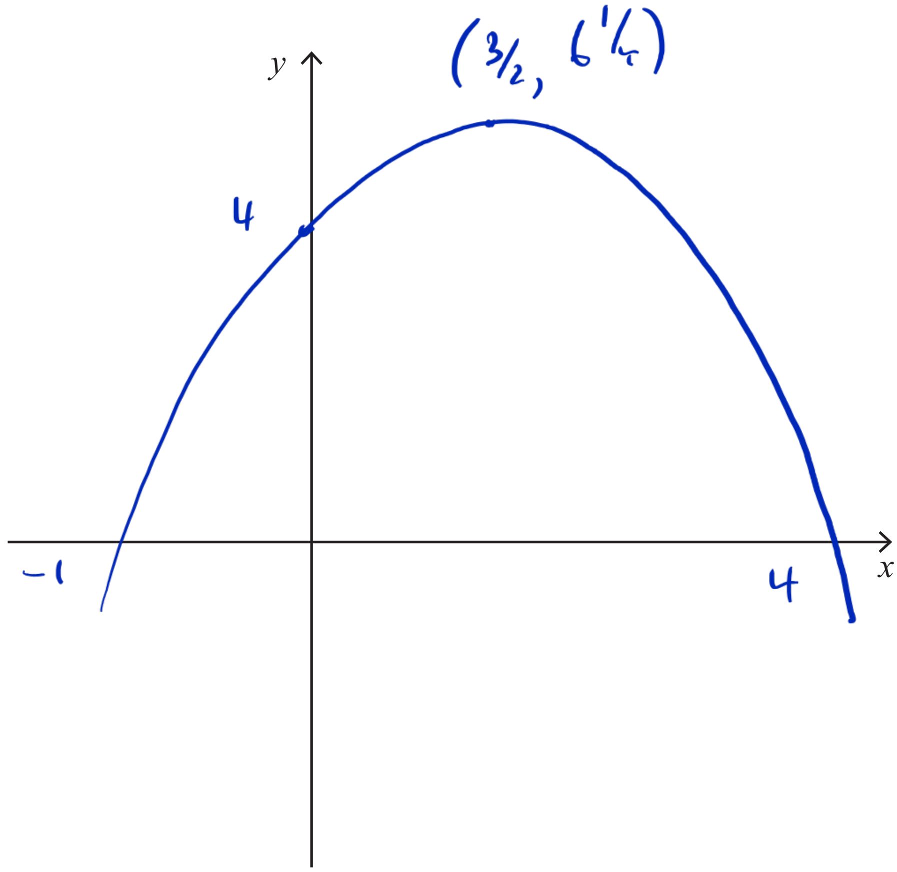
Answer $(3/2, 6\frac{1}{4})$ [4]

(iii) Using calculus, identify the turning point as either a maximum or a minimum point. You **must** show working to justify your answer.

$\frac{d^2y}{dx^2} = -2 \Rightarrow -ve \quad \cap \quad$ MAX

Answer MAX [1]

(iv) Sketch the curve on the axes below.



[2]

(v) Find the area enclosed by the curve and the x-axis.

$$\int_{-1}^{4} (-x^2 + 3x + 4)\, dx$$

$$\left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x \right]_{-1}^{4}$$

$$\left[-\frac{(4)^3}{3} + \frac{3(4)^2}{2} + 4(4) \right] - \left[-\frac{(-1)^3}{3} + \frac{3(-1)^2}{2} + 4(-1) \right]$$

$$\left[\frac{56}{3} \right] - \left[-\frac{13}{6} \right]$$

Answer $20\frac{5}{6}$ [4]

10 Matrices **A**, **B** and **C** are defined by

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 \\ 7 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Find the matrix **X** such that

$$\mathbf{AX} = \mathbf{B} + \mathbf{C}$$

$$A^{-1}AX = A^{-1}(B+C)$$

$$\downarrow$$

$$A^{-1} \rightarrow \frac{1}{2}\begin{bmatrix} -5 & 3 \\ -4 & 2 \end{bmatrix} \qquad \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} -5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -5(-3) + 3(5) \\ -4(-3) + 2(5) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 30 \\ 22 \end{bmatrix} = \begin{bmatrix} 15 \\ 11 \end{bmatrix}$$

Answer ______________________________ [5]

11 An airline operates two flights each week from Belfast to Lanzarote.

The airline offers three types of fare

- a premium fare of £70, $70x$
- an economy fare of £55 and $55y$
- a stand-by fare of £35 $35z$

Let x, y and z represent the numbers of passengers paying premium, economy and stand-by fares respectively on the first flight of a particular week.

On this flight all 180 seats were sold.

Hence x, y and z satisfy the equation

$$x + y + z = 180$$

The total income from fares for this flight was £9325

(i) Show that x, y and z satisfy the equation

$$14x + 11y + 7z = 1865$$

$70x + 55y + 35z = 9325$

$\div 5$ ( ) $\div 5$

$14x + 11y + 7z = 1865$

[1]

Compared with the first flight, the second flight had

- $\frac{2}{3}$ the number of passengers paying the premium fare, $\frac{2}{3}x$
- 25 more passengers paying the economy fare and $(y+25)$
- half the number of passengers paying the stand-by fare. $\frac{z}{2}$

Again, all 180 seats were sold.

(ii) Show that x, y and z also satisfy the equation

$$4x + 6y + 3z = 930$$

$$\frac{2}{3}x + (y+25) + \frac{z}{2} = 180$$

$$\frac{2}{3}x + y + \frac{z}{2} = 155$$

$\times 3$ $\times 2$ $\times 3$ $\times 2$

$$4x + 6y + 3z = 930$$

[2]

(iii) Solve the equations below to find the numbers of passengers paying each type of fare on the **first** flight, showing clearly each stage of your solution.

$$\begin{aligned} x + y + z &= 180 \quad A \\ 14x + 11y + 7z &= 1865 \quad B \\ 4x + 6y + 3z &= 930 \quad C \end{aligned}$$

B − 7A

$14x + 11y + 7z = 1865$

$7x + 7y + 7z = 1260$

$7x + 4y = 605$ D

C − 3A

$4x + 6y + 3z = 930$

$3x + 3y + 3z = 540$

$x + 3y = 390$ E

7E − D

$7x + 21y = 2730$

$7x + 4y = 605$

$17y = 2125$

$y = 125$

$x + 3y = 390$

$x = 390 - 3(125)$

$x = 15$

$x + y + z = 180$

$z = 180 - (125 + 15)$

$= 40$

Answer Number paying premium 15

Number paying economy 125

Number paying stand-by 40 [8]

(iv) Hence find the total income from fares for the **second** flight.

$\frac{2}{3}x \rightarrow 10 \rightarrow 10 \times £70$

$y+25 \rightarrow 150 \rightarrow 150 \times £55$

$z \div 2 \rightarrow 20 \rightarrow 20 \times £35$

£9650

Answer £ 9650 [2]

[Turn over

12 A curve is defined by the equation

$$y = \frac{2}{3}x^2(x-3)$$

(i) Find the equation of the tangent to the curve at the point $(1, -1\frac{1}{3})$.

$$y = \frac{2}{3}x^3 - 2x^2$$

gradient tangent

sub in $x=1$

$$\frac{dy}{dx} = 2x^2 - 4x$$

$$\frac{dy}{dx} = 2(1)^2 - 4(1)$$

$$\frac{dy}{dx} = -2$$

$$y = mx + c$$

$$-1\tfrac{1}{3} = -2(1) + c$$

$$c = \tfrac{2}{3}$$

Answer $y = -2x + \frac{2}{3}$ [3]

(ii) Find the equation of the normal to the curve at the point $(-1,\ -2\frac{2}{3})$.

tangent → $\frac{dy}{dy} = 2x^2 - 4x$

$= 2(-1)^2 - 4(-1)$

$= 6$

normal → $m = -1/6$

$y = mx + c$

$-2\,2/3 = -1/6\,(-1) + c$

$c = -2\,5/6$

Answer $y = -1/6\,x - 2\,5/6$ [2]

Q12 continues on page 25

[Turn over

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Q12 continues on opposite page

(iii) Hence find the **exact** value of the x-coordinate at the point of intersection of the tangent in **(i)** and the normal in **(ii)**.

$y = -\frac{1}{6}x - 2\frac{5}{6}$

$y = -2x + \frac{2}{3}$

$-\frac{1}{6}x - 2\frac{5}{6} = -2x + \frac{2}{3}$

$-\frac{1}{6}x + 2x = 2\frac{5}{6} + \frac{2}{3}$

$1\frac{5}{6}x = \frac{7}{2}$

$x = \frac{21}{11}$

Answer $\frac{21}{11}$ [2]

13 The populations of various towns and cities in Northern Ireland were recorded in 2001. They were put in rank order according to their population.

The table below shows the rank, R, and the population, P, of the second to the sixth largest population centres.

Rank R	**Population** P	x $\log R$	y $\log P$
2	85 000	x_1 0.301	y_1 4.929
3	71 000	0.477	4.851
4	63 000	0.602	4.799
5	57 000	0.699	4.756
6	52 000	x_2 0.778	y_2 4.716

Martin believes that a relationship of the form

For $m \rightarrow \dfrac{y_2 - y_1}{x_2 - x_1}$

$$P = aR^n$$

exists for ranks 2 to 6, where a and n are constants.

$$P = aR^n$$

$$\log P = \log aR^n$$

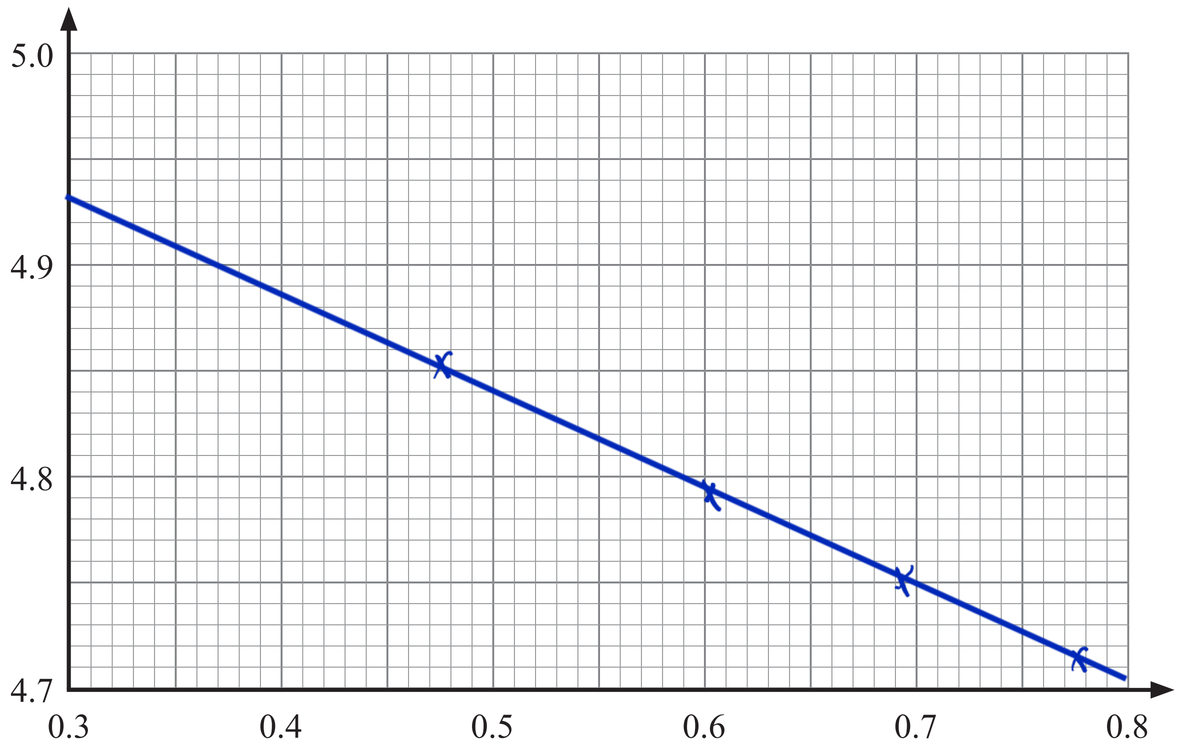
$$\log P = n \log R + \log a$$

$$y = mx + c$$

(i) Verify that a relationship of the form $P = aR^n$ exists by drawing a suitable straight line graph on the grid below.

Label the axes clearly.

Show clearly the values used, correct to 3 decimal places, in the table opposite.



[6]

Q13 continues on page 29

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(ii) **Hence** find the value of n, correct to 2 decimal places, and the value of a, correct to the nearest integer.

$n = \dfrac{4.716 - 4.929}{0.778 - 0.301} = -0.44654$

$P = aR^n$

$85000 = a2^{-0.44654}$

$\left(\begin{array}{l} R=2 \\ P=85000 \end{array}\right)$

Answer n = -0.45, a = 115 835 [4]

The largest population centre in 2001 was Belfast with a population of 333 000

(iii) Show clearly, using your values for a and n, that the relationship $P = aR^n$ does **not** hold for Belfast.

Rank = 1

$P = aR^n$

$P = 115\,835\,(1)^{-0.45}$

$= 115\,835$

not approx 333 000

so does not hold

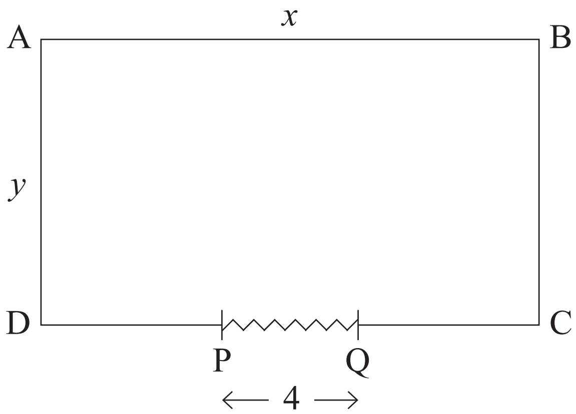
[1]

14 A rancher wishes to build a rectangular enclosure ABCD for horses.

He has 100 m of fencing to construct the perimeter of the enclosure.

He plans to use a metal gate PQ, of length 4 m, for the entrance to the enclosure.

He plans to use all the fencing for the rest of the perimeter.



Let x and y be the length and width, in metres, of the enclosure.

(i) Derive an expression for y in terms of x.

$y + x + y + (x - 4) = 100$

$2y + 2x - 4 = 100$

$2y = 104 - 2x$

$y = 52 - x$

Answer $y =$ ____________________ [2]

(ii) **Hence** show that the area of the enclosure is given by

$$A = 52x - x^2$$

Area $= xy$

$= x(52 - x)$

$= 52x - x^2$

[1]

(iii) Find the dimensions of the enclosure which will give the maximum area, proving that it is a maximum.

$\frac{dA}{dx} = 52 - 2x = 0$

$2x = 52$

$x = 26$

$y = 52 - x =$

$\frac{d^2A}{dx^2} = -2 \Rightarrow \cap \Rightarrow$ MAX

Answer Length 26 m

Width 26 m [4]

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For Examiner's use only	
Question Number	Marks
1	
2	
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Total Marks	

Examiner Number

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